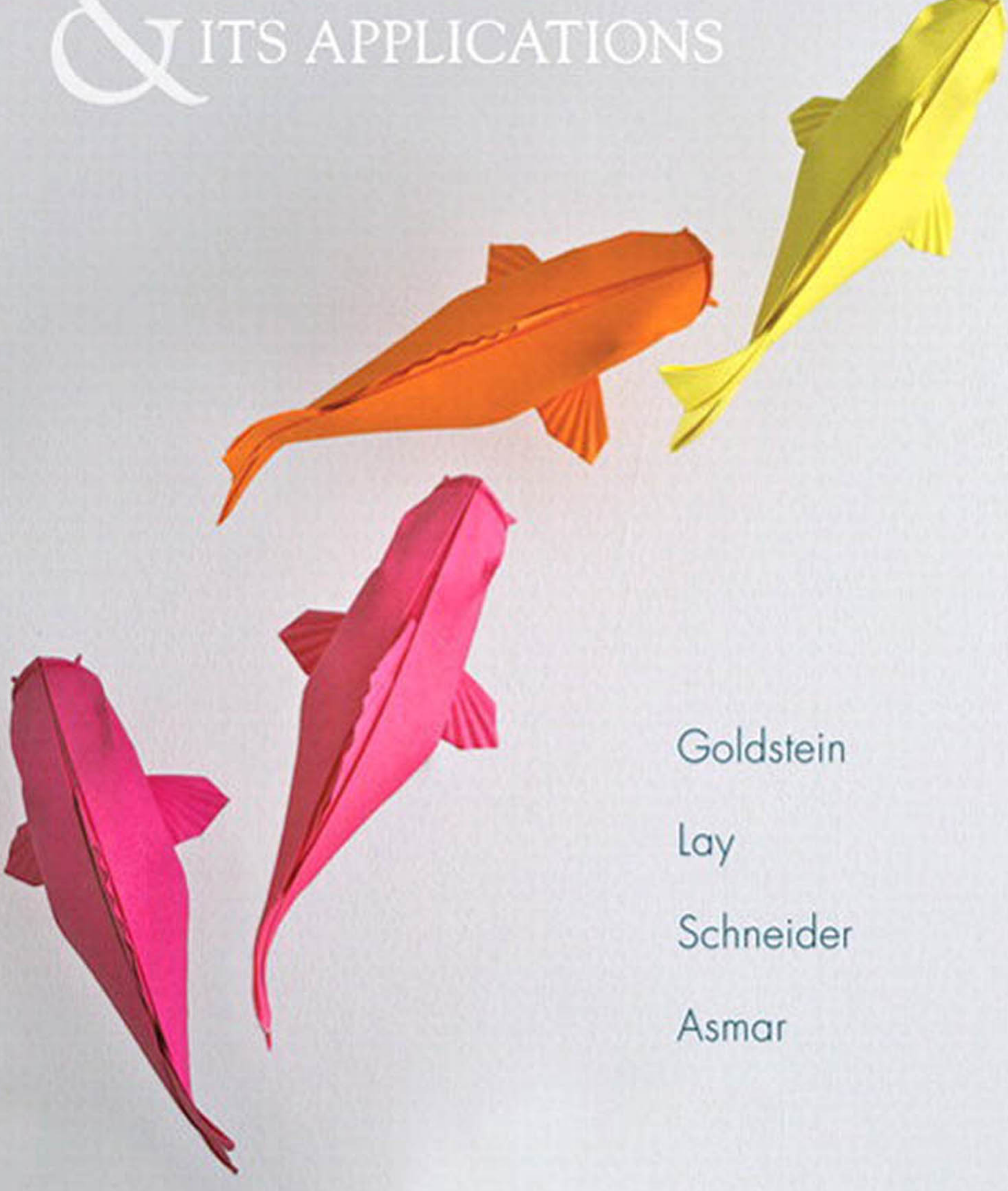
Thirteenth Edition

CALCULUS & ITS APPLICATIONS



Goldstein

Lay

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Thirteenth Edition

CALCULUS & ITS APPLICATIONS

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Contents

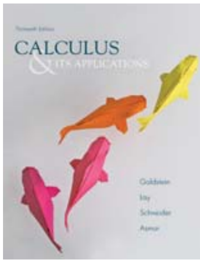
Preface

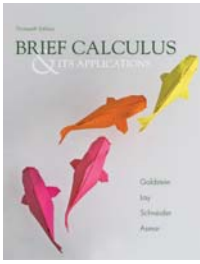
This thirteenth edition of *Calculus and Its Applications*, and its Brief version, is written for either a one- or two-semester applied calculus course. Although this edition reflects many revisions as requested by instructors across the country, the foundation and approach of the text has been preserved. In addition, the level of rigor and flavor of the text remains the same. Our goals for this revision reflect the original goals of the text which include: to begin calculus as soon as possible; to present calculus in an intuitive yet intellectually satisfying way; and to integrate the many applications of calculus to business, life sciences, and social sciences.

This proven approach, as outlined below, coupled with newly updated applications, the integration of tools to make the calculus more accessible to students, and a greatly enhanced MyMathLab course, make this thirteenth edition a highly effective resource for your applied calculus courses.

The Series

This text is part of a highly successful series consisting of three texts: *Finite Mathematics and Its Applications*, *Calculus and Its Applications*, and *Brief Calculus and Its Applications*. All three titles are available for purchase as a printed text, an eBook within the MyMathLab online course, or both.





Topics Included

The distinctive order of topics has proven over the years to be successful. The presentation of topics makes it easier for students to learn, and more interesting because students see significant applications early in the course. For instance, the derivative is explained geometrically before the analytic material on limits is presented. To allow you to reach the applications in Chapter 2 quickly, we present only the differentiation rules and the curve sketching needed.

Because most courses do not afford enough time to cover all the topics in this text and because different schools have different goals for the course, we have been strategic with the placement and organization of topics. To this end, the level of theoretical material may be adjusted to meet the needs of the students. For example, Section 1.4 may be omitted entirely if the instructor does not wish to present the notion of limit beyond the required material that is contained in Section 1.3. In addition, sections considered optional are starred in the table of contents.

Prerequisites

Because students often enter this course with a variety of prerequisite skills, Chapter 0 is available to either cover in its entirety or as a source for remediation depending on the pace of the course. In addition to being covered in Chapter 0, some important topics, such as the laws of exponents, are reviewed again when they are used in a later chapter.

New to this edition, we have added a **Prerequisite Skills Diagnostic Test** prior to Chapter 0 so students or instructors can assess weak areas. The answers to the diagnostic test are provided in the student edition answer section along with references to areas where students can go for remediation. Remediation is also available within MyMathLab through the newly created Getting Ready for Applied Calculus content at the start of select chapters.

New to This Edition

This text has been refined and improved over the past twelve editions via the many instructor recommendations, student feedback, and years of author experience. However, there are always improvements to be made in the clarity of the exposition, the relevance of the applications and the quality of exercise sets. To this end, the authors have worked diligently to fine-tune the presentation of the topics, update the applications, and improve the gradation and thoroughness of the exercise sets throughout the text. In addition, there are a few topics that the authors focused on to better enhance the learning experience for students.

- **The Derivative (Chapter 1)** In the previous edition, the derivative is introduced in Section 1.3 in an intuitive way, using examples of slopes of tangent lines and applied problems involving rates of change from Section 1.2. In the current edition, Section 1.3 incorporates an intuitive introduction to limits, as they arise from the computations of derivatives. This approach to limits paves the way to the more detailed discussion on limits in Section 1.4. It offers the instructor the option of spending less time on limits (and therefore more time on the application) by not emphasizing or completely skipping Section 1.4.
- **The Integral (Chapter 6)** Chapter 6 has been significantly reworked. As in the previous edition, we introduce the antiderivative in Section 6.1. However, we have simplified the presentation in Section 6.1 by opening with an example involving the velocity and position functions of a moving object. This example motivates the introduction of the antiderivative or *indefinite integral* in a natural way. Section 6.2 builds on the momentum from the examples of antiderivatives and introduces the *definite integral* using the formula

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

  where F is an antiderivative of f. Several new examples are presented in Section 6.2 that illustrate the importance of the definite integral and the use of the antiderivative (net change in position, marginal revenue analysis, net increase in federal health expenditures).

  In Section 6.3, we introduce the concepts of Riemann sums and areas of regions under a graph, and prove the Fundamental Theorem of Calculus by showing that the Riemann sums converge to the definite integral. Our new approach allows for an easier flow of the discussion of integration by moving directly from the indefinite integral to the definite integral, without the diversion into Riemann sums. While the classical approach to the definite integral (as a limit of Riemann sums) has the concepts of limits and area as a driver, our approach emphasizes the applications as the driver for the definite integral. More importantly, our new approach allows students to compute areas using basic geometric formulas (areas of rectangles and right triangles) and compare their results to those obtained by using the definite integral. In contrast to the approach based on limits of Riemann sums, our approach provides a hands-on approach to areas and brings students closer to understanding the concept of Riemann sums and areas. (See the Introduction and Example 1 in Section 6.3.)

- **Prerequisite Skills Diagnostic Test** A quick assessment of the basic algebra skills students should already have mastered is provided prior to Chapter 0. This can be a self-diagnostic tool for students, or instructors can use it to gain a sense of where review for the entire class may be helpful. Answers are provided in the back of the student edition along with references to where students can go for remediation within the text.
- **Now Try Exercises** Students are now given Now Try exercises to encourage an immediate check of their understanding of a given example by solving a specific, odd-numbered exercise from the exercise sets.
- **Additional Exercises and Updated Applications** We have added many new exercises and have updated the real-world data appearing in the examples and exercises whenever possible.
- **Chapter Summaries** Each chapter ends with a summary that directs students to the important topics in the sections. In addition, we have identified topics that may be challenging to students and presented several helpful examples. Most notably, we have added examples that illustrate differentiation and integration rules, integration by parts, solving optimization problems, setting up equations arising from modeling, solving problems involving functions of more than one variable (Lagrange multiplier, second derivative test in two dimensions), and differential equations, to name just a few. Effectively, the summaries contain more than one hundred additional, completely worked examples.
- **Answers to Fundamental Concept Check Exercises** We have added the answers to the Fundamental Concept Check Exercises so students can check their understanding of the main concepts in each chapter.
- **Chapter Objectives** The key learning objectives for each section of the text are enumerated in the back of the text. These objectives will be especially helpful for instructors who need to verify that particular skills are covered in the text.
- **Summary Endpapers** A two-page spread at the back of the text lists key definitions, theorems and formulas from the course for an easy reference guide for students.

New Resources Outside the Text

1. **Annotated Instructor's Edition** New to this edition, an annotated instructor's edition is available to qualified adopters of the text. The AIE is a highly valuable resource for instructors with answers to the exercises on the same page as the exercise, whenever possible, making it easier to assign homework based on the skill level and interests of each class. Teaching Tips are also provided in the AIE margins to highlight for new instructors the common pitfalls made by students.
2. **Updates to MyMathLab (MML)** Many improvements have been made to the overall functionality of MML since the previous edition. However, beyond that, we have also invested greatly in increasing and improving the content specific to this text.

   a. Instructors now have **more exercises** to choose from in assigning homework.
   b. An extensive evaluation of individual exercises in MML has resulted in minor edits and refinements making for an **even stronger connection** between the exercises available in the text with the exercises available in MML.
   c. **Interactive Figures** have been developed specifically for this text as a way to provide students with a visual representation of the mathematics. These interactive figures are integrated into the eText for student use, available in the instructor's resources as a presentational tool, and are tied to specifically designed exercises in MyMathLab for targeted instruction and assessment. The Interactive Figures run on the freely available Wolfram CDF Player.

d. **Application exercises** in MML are now labeled with the field of study to which they relate. This allows instructors to design online homework assignments more specific to the students' fields of study (i.e., Bus Econ; Life Sci; Social Sci; Gen Interest)
e. Because students' struggles with prerequisite algebra skills are the most serious roadblock to success in this course, we have added **"Getting Ready" content** to the beginning of chapters in MML. These introductory sections contain the prerequisite algebra skills critical for success in that chapter. Instructors can either sprinkle these exercises into homework assignments as needed or use MML's built-in diagnostic tests to assess student knowledge of these skills, then automatically assign remediation for only those skills that students have not mastered (using the Personalized Homework functionality). This assessment tied to personalized remediation provides an extremely valuable tool for instructors, and help "just in time" for students.

Trusted Features

Though this edition has been improved in a variety of ways to reflect changing student needs, we have maintained the popular overall approach that has helped students be successful over the years.

Relevant and Varied Applications

We provide realistic applications that illustrate the uses of calculus in other disciplines and everyday life. The reader may survey the variety of applications by referring to the Index of Applications at the end of the text. Wherever possible, we attempt to use applications to motivate the mathematics. For example, our approach to the derivative in Section 1.3 is motivated by the slope formula and applications in Section 1.2, and applications of the net change of functions in Section 6.2 motivate our approach to the integral in 6.3.

Plentiful Examples

The text provides many more worked examples than is customary. Furthermore, we include computational details to enhance comprehension by students whose basic skills are weak. Knowing that students often refer back to examples for help as they work through exercises, we closely reviewed the fidelity between exercises and examples in this revision, making adjustments as necessary to create a better resource for the students.

Exercises to Meet All Student Needs

The exercises comprise about one-quarter of the book, the most important part of the text, in our opinion. The exercises at the ends of the sections are typically arranged in the order in which the text proceeds, so that homework assignments may be made easily after only part of a section is discussed. Interesting applications and more challenging problems tend to be located near the ends of the exercise sets. An additional effort has been made in this edition to create an even stronger odd-even pairing between exercises, when appropriate. **Chapter Review Exercises** at the end of each chapter amplify the other exercise sets and provide cumulative exercises that require skills acquired from earlier chapters. Answers to the odd-numbered exercises, and all Chapter Review Exercises, are included at the back of the book.

Check Your Understanding Problems

The **Check Your Understanding Problems**, formerly called Practice Problems, are a popular and useful feature of the book. They are carefully selected exercises located at the end of each section, just before the exercise set. Complete solutions follow the exercise set. These problems prepare students for the exercise sets beyond just covering prerequisite skills or simple examples. They give students a chance to think about the skills they are about to apply and reflect on what they've learned.

Use of Technology

As in previous editions, the use of graphing calculators is not required for the study of this text; however graphing calculators are very useful tools that can be used to simplify computations, draw graphs, and sometimes enhance understanding of the fundamental topics of calculus. Helpful information about the use of calculators appears at the end of most sections in subsections titled **Incorporating Technology**. The examples have been updated to illustrate the use of the family of TI-83/84 calculators, in particular, most screenshots display in MathPrint mode from the new TI-84+ Silver Edition.

End-of-Chapter Study Aids

Near the end of each chapter is a set of problems entitled Fundamental Concept Check Exercises that help students recall key ideas of the chapter and focus on the relevance of these concepts as well as prepare for exams. The answers to these exercises are now included in the back-of-book student answer section. Each chapter also contains a new two-column grid giving a section-by-section summary of key terms and concepts with examples. Finally, each chapter has Chapter Review Exercises that provide more practice and preparation for chapter-level exams.

Supplements

For Students

Student's Solutions Manual
(ISBN: 0-321-87857-4/978-0-321-87857-1)
Contains fully worked solutions to odd-numbered exercises.

Video Lectures with Optional Captioning (online)
These comprehensive, section-level videos provide excellent support for students who need additional assistance, for self-paced or online courses, or for students who missed class. The videos are available within MyMathLab at both the section level, as well as at the example level within the eText.

For Instructors

Annotated Instructor's Edition
(ISBN: 0-321-86461-1/978-0-321-86461-1)
New to this edition, the Annotated Instructor's Edition provides answers to the section exercises on the page whenever possible. In addition, Teaching Tips provide insightful comments to those who are new to teaching the course.

Instructor's Solutions Manual (downloadable)
Includes fully worked solutions to every textbook exercise. Available for download for qualified instructors within MyMathLab or through the Pearson Instructor Resource Center, www.pearsonhighered.com/irc.

PowerPoint Lecture Presentation
Contains classroom presentation slides that are geared specifically to the textbook and contain lecture content and key graphics from the book. Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center, www.pearsonhighered.com/irc.

TestGen® (downloadable)
TestGen (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and testbank are available for download from Pearson Education's online catalog, www.pearsonhighered.com/irc.

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MyMathLab provides **engaging experiences** that personalize, stimulate, and measure learning for each student.

- **Exercises:** The homework and practice exercises in MyMathLab are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. The software offers immediate, helpful feedback when students enter incorrect answers.
- **Multimedia Learning Aids:** Exercises include guided solutions, sample problems, animations, videos, interactive figures, and eText access for extra help at point of use.

- **Getting Ready Content:** Prerequisite skills content is now provided chapter by chapter within MyMathLab. You can include this content as needed within regular homework assignments or use MyMathLab's built-in diagnostic tests to assess gaps in skills and automatically assign remediation to address those gaps.
- **Expert Tutoring:** Although many students describe the whole of MyMathLab as "like having your own personal tutor," students using MyMathLab do have access to live tutoring from Pearson, from qualified math and statistics instructors.

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- **Login from Your Campus Portal:** You and your students can link directly from your campus portal into your MyLabsPlus courses. A Pearson service team works with your institution to create a single sign-on experience for instructors and students.
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MathXL is the homework and assessment engine that runs MyMathLab. (MyMathLab is MathXL plus a learning management system.)
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- Create and assign their own online exercises and import TestGen tests for added flexibility.
- Maintain records of all student work tracked in MathXL's online gradebook.

With MathXL, students can

- Take chapter tests in MathXL and receive personalized study plans and/or personalized homework assignments based on their test results.
- Use the study plan and/or the homework to link directly to tutorial exercises for the objectives they need to study.
- Access supplemental animations and video clips directly from selected exercises.

MathXL is available to qualified adopters. For more information, visit our website at www.mathxl.com or contact your Pearson representative.

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If you have any comments or suggestions, we would like to hear from you. We hope you enjoy using this book as much as we have enjoyed writing it.

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Prerequisite Skills Diagnostic Test

To the Student and the Instructor Are you ready for calculus? This prerequisite skills diagnostic test evaluates basic mathematical skills that are required to begin the course. It is not intended to replace a placement test that your institution may already have. Each set of questions refers to a section in Chapter 0 of the text. If you miss several questions from one part, you may want to study the corresponding section from Chapter 0.

Calculate the given quantities using the laws of exponents. **(Section 0.5)**

1. $\dfrac{7^{3/2}}{49}\sqrt{7}$

2. $(3^{1/3}3^{1/2})^2\sqrt[3]{3}$

3. $\dfrac{\sqrt{5}}{\sqrt{15}\sqrt{3}}$

4. $2^{1/3}2^{1/2}2^{1/6}$

Simplify the given expressions. Your answer should not involve parentheses or negative exponents. **(Section 0.5)**

5. $\dfrac{x^2}{x^{-4}}$

6. $\dfrac{x^2(x^{-4}+1)}{x^{-2}}$

7. $\left(\dfrac{x}{x^2y^2}\right)^3 y^8$

8. $\left(\dfrac{1}{xy}\right)^{-2}\left(\dfrac{x}{y}\right)^2$

Given $f(x) = \dfrac{x}{x+1}$ *and* $g(x) = x+1$, *express the following as a rational functions.* **(Section 0.3)**

9. $f(x) + g(x)$

10. $f(x)g(x)$

11. $\dfrac{g(x)}{f(x)}$

12. $f(x) - \dfrac{g(x)}{x+1}$

Given $f(t) = t^2$ *and* $g(t) = \dfrac{t}{t+1}$, *calculate the following functions. Simplify your answer as much as possible.* **(Section 0.3)**

13. $f(g(t))$

14. $g(f(t))$

15. $f(f(g(t)))$

16. $f(g(t+1))$

For the given $f(x)$, *find* $\dfrac{f(x+h) - f(x)}{h}$ *and simplify your answer as much as possible.* **(Section 0.3)**

17. $f(x) = x^2 + 2x$

18. $f(x) = \dfrac{1}{x}$

19. $f(x) = \sqrt{x}$ (*Hint:* Rationalize the numerator.)

20. $f(x) = x^3 - 1$

Graph the following functions. Determine clearly the intercepts of the graphs. **(Section 0.2)**

21. $f(x) = 2x - 1$

22. $f(x) = -x$

23. $f(x) = -\dfrac{x-1}{2}$

24. $f(x) = 3$

Find the points of intersection (if any) of the pairs of curves. **(Section 0.4)**

25. $y = 3x + 1,\ y = -x - 2$

26. $y = \dfrac{x}{2},\ y = 3$

27. $y = 4x - 7,\ y = 0$

28. $y = 2x + 3,\ y = 2x - 2$

Factor the given polynomials. **(Section 0.4)**

29. $x^2 + 5x - 14$

30. $x^2 + 5x + 4$

31. $x^3 + x^2 - 2x$

32. $x^3 - 2x^2 - 3x$

Solve the given equations by factoring first. **(Section 0.4)**

33. $x^2 - 144 = 0$

34. $x^2 + 4x + 4 = 0$

35. $x^3 + 8x^2 + 15x = 0$

36. $6x^3 + 11x^2 + 3x = 0$

Solve using the quadratic formula. **(Section 0.4)**

37. $2x^2 + 3x - 1 = 0$

38. $x^2 + x - 1 = 0$

39. $-3x^2 + 2x - 4 = 0$

40. $x^2 + 4x - 4 = 0$

Solve the given equations. **(Section 0.4)**

41. $x^2 - 3x = 4x - 10$

42. $4x^2 + 2x = -2x + 3$

43. $\dfrac{1}{x+1} = x + 1$

44. $\dfrac{x^3}{x^2 + 2x - 1} = x - 1$

Introduction

> *"Calculus provides mathematical tools to study each change in a quantitative way."*

Often, it is possible to give a succinct and revealing description of a situation by drawing a graph. For example, Fig. 1 describes the amount of money in a bank account drawing 5% interest, compounded daily. The graph shows that, as time passes, the amount of money in the account grows. Figure 2 depicts the weekly sales of a breakfast cereal at various times after advertising has ceased. The graph shows that the longer the time since the last advertisement, the fewer the sales. Figure 3 shows the size of a bacteria culture at various times. The culture grows larger as time passes. But there is a maximum size that the culture cannot exceed. This maximum size reflects the restrictions imposed by food supply, space, and similar factors. The graph in Fig. 4 describes the decay of the radioactive isotope iodine 131. As time passes, less and less of the original radioactive iodine remains.

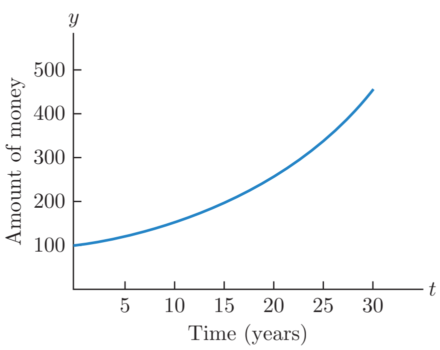


Figure 1 Growth of money in a savings account.

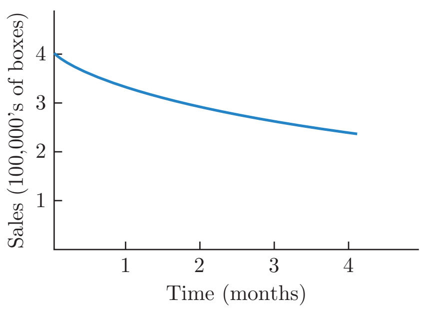


Figure 2 Decrease in sales of breakfast cereal.

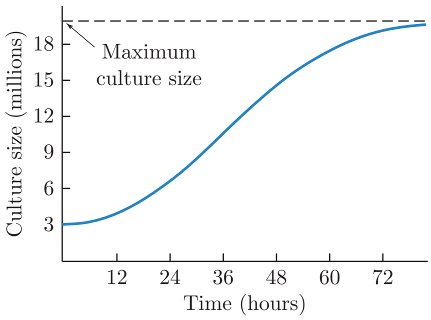


Figure 3 Growth of a bacteria culture.

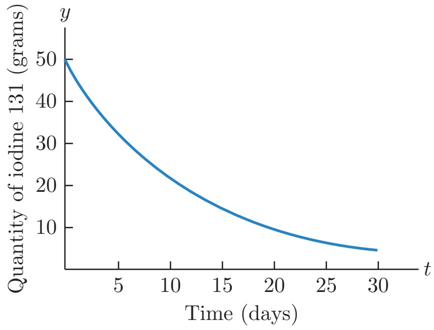


Figure 4 Decay of radioactive iodine.

Each graph in Figs. 1 to 4 describes a change that is taking place. The amount of money in the bank is changing, as are the sales of cereal, the size of the bacteria culture, and the amount of iodine. Calculus provides mathematical tools to study each change in a quantitative way.

CHAPTER 0

Functions

Each graph in Figs. 1 to 4 of the Introduction depicts a relationship between two quantities. For example, Fig. 4 illustrates the relationship between the quantity of iodine (measured in grams) and time (measured in days). The basic quantitative tool for describing such relationships is a *function*. In this preliminary chapter, we develop the concept of a function and review important algebraic operations on functions used later in the text.

0.1 Functions and Their Graphs

Real Numbers

Most applications of mathematics use real numbers. For purposes of such applications (and the discussions in this text), it suffices to think of a real number as a decimal. A *rational* number is one that may be written as a finite or infinite repeating decimal or as a fraction, such as

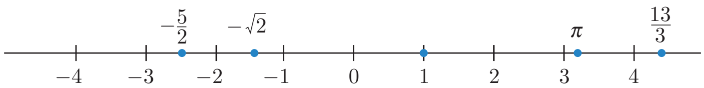
$$-\frac{5}{2} = -2.5, \qquad 1, \qquad \frac{13}{3} = 4.333\ldots \qquad \text{(rational numbers)}.$$

An *irrational* number has an infinite decimal representation whose digits form no repeating pattern, such as

$$-\sqrt{2} = -1.414213\ldots, \qquad \pi = 3.14159\ldots \qquad \text{(irrational numbers)}.$$

The real numbers are described geometrically by a *number line*, as in Fig. 1. Each number corresponds to one point on the line, and each point determines one real number.

Figure 1 The real number line.



We use four types of inequalities to compare real numbers.

$x < y$	x is less than y
$x \leq y$	x is less than or equal to y
$x > y$	x is greater than y
$x \geq y$	x is greater than or equal to y

The double inequality $a < b < c$ is shorthand for the pair of inequalities $a < b$ and $b < c$. Similar meanings are assigned to other double inequalities, such as $a \leq b < c$. Three numbers in a double inequality, such as $1 < 3 < 4$ or $4 > 3 > 1$, should have the same relative positions on the number line as in the inequality (when read left to right or right to left). Thus $3 < 4 > 1$ is never written because the numbers are "out of order."

Geometrically, the inequality $x \leq b$ means that either x equals b or x lies to the left of b on the number line. The set of real numbers x that satisfies the double inequality $a \leq x \leq b$ corresponds to the line segment between a and b, including the endpoints. This set is sometimes denoted by $[a, b]$ and is called the *closed interval* from a to b. If a and b are removed from the set, the set is written as (a, b) and is called the *open interval* from a to b. The notation for various line segments is listed in Table 1.

TABLE 1 Intervals on the Number Line

Inequality	Geometric Description	Interval Notation
$a \leq x \leq b$		$[a, b]$
$a < x < b$		(a, b)
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$a \leq x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

The symbols ∞ ("infinity") and $-\infty$ ("minus infinity") do not represent actual real numbers. Rather, they indicate that the corresponding line segment extends infinitely far to the right or left. An inequality that describes such an infinite interval may be written in two ways. For instance, $a \leq x$ is equivalent to $x \geq a$.

EXAMPLE 1 **Graphing Intervals** Describe each of the following intervals both graphically and in terms of inequalities.

(a) $(-1, 2)$ **(b)** $[-2, \pi]$ **(c)** $(2, \infty)$ **(d)** $(-\infty, \sqrt{2}]$

SOLUTION The line segments corresponding to the intervals are shown in Figs. 2(a)–(d). Note that an interval endpoint that is included (e.g., both endpoints of $[a, b]$) is drawn as a solid circle, whereas an endpoint not included (e.g., the endpoint a in $(a, b]$) is drawn as an unfilled circle.

EXAMPLE 2 **Using Inequalities** The variable x describes the profit that a company anticipates earning in the current fiscal year. The business plan calls for a profit of at least 5 million dollars. Describe this aspect of the business plan in the language of intervals.

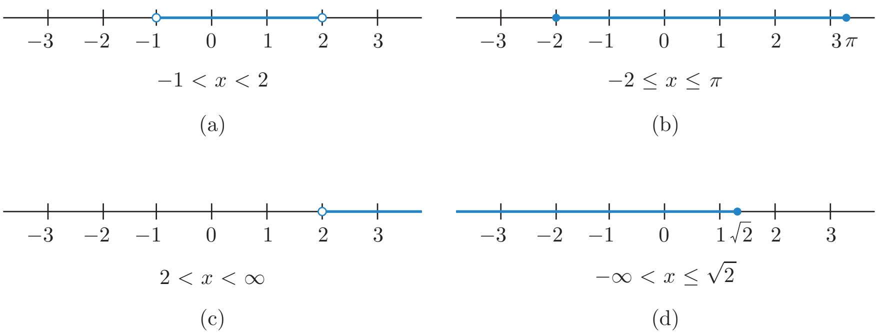


Figure 2 Line segments.

SOLUTION The phrase "at least" means "greater than or equal to." The business plan requires that $x \ge 5$ (where the units are millions of dollars). This is equivalent to saying that x lies in the infinite interval $[5, \infty)$.

Functions A *function* of a variable x is a *rule* f that assigns to each value of x a unique number $f(x)$, called *the value of the function at* x. [We read "$f(x)$" as "f of x."] The variable x is called the *independent variable.* The set of values that the independent variable is allowed to assume is called the *domain* of the function. The domain of a function may be explicitly specified as part of the definition of a function, or it may be understood from context. (See the following discussion.) The *range* of a function is the set of values that the function assumes.

The functions we shall meet in this book will usually be defined by algebraic formulas. For example, the domain of the function

$$f(x) = 3x - 1$$

consists of all real numbers x. This function is the rule that takes a number, multiplies it by 3, and then subtracts 1. If we specify a value of x—say, $x = 2$—then we find the value of the function at 2 by substituting 2 for x in the formula:

$$f(2) = 3(2) - 1 = 5.$$

EXAMPLE 3 **Evaluating a Function** Let f be the function with domain all real numbers x and defined by the formula

$$f(x) = 3x^3 - 4x^2 - 3x + 7.$$

Find $f(2)$ and $f(-2)$.

SOLUTION To find $f(2)$, we substitute 2 for every occurrence of x in the formula for $f(x)$:

$$\begin{aligned} f(2) &= 3(2)^3 - 4(2)^2 - 3(2) + 7 \\ &= 3(8) - 4(4) - 3(2) + 7 \\ &= 24 - 16 - 6 + 7 \\ &= 9. \end{aligned}$$

To find $f(-2)$, we substitute (-2) for each occurrence of x in the formula for $f(x)$. The parentheses ensure that the -2 is substituted correctly. For instance, x^2 must be replaced by $(-2)^2$, not -2^2:

$$\begin{aligned} f(-2) &= 3(-2)^3 - 4(-2)^2 - 3(-2) + 7 \\ &= 3(-8) - 4(4) - 3(-2) + 7 \\ &= -24 - 16 + 6 + 7 \\ &= -27. \end{aligned}$$

▶ ***Now Try Exercise 13***

EXAMPLE 4 **Temperature Scales** If x represents the temperature of an object in degrees Celsius, then the temperature in degrees Fahrenheit is a function of x, given by $f(x) = \frac{9}{5}x + 32$.

(a) Water freezes at 0°C (C = Celsius) and boils at 100°C. What are the corresponding temperatures in degrees Fahrenheit (F = Fahrenheit)?

(b) Aluminum melts at 660°C. What is its melting point in degrees Fahrenheit?

SOLUTION **(a)** $f(0) = \frac{9}{5}(0) + 32 = 32$. Water freezes at 32°F.
$f(100) = \frac{9}{5}(100) + 32 = 180 + 32 = 212$. Water boils at 212°F.

(b) $f(660) = \frac{9}{5}(660) + 32 = 1188 + 32 = 1220$. Aluminum melts at 1220°F.

In the preceding examples, the functions had domains consisting of all real numbers or an interval. For some functions, the domain may consist of several intervals, with a different formula defining the function on each interval. Here is an illustration of this phenomenon.

EXAMPLE 5 **A Piecewise-Defined Function** A leading brokerage firm charges a 6% commission on gold purchases in amounts from \$50 to \$300. For purchases exceeding \$300, the firm charges 2% of the amount purchased plus \$12.00. Let x denote the amount of gold purchased (in dollars) and let $f(x)$ be the commission charge as a function of x.

(a) Describe $f(x)$.

(b) Find $f(100)$ and $f(500)$.

SOLUTION **(a)** The formula for $f(x)$ depends on whether $50 \le x \le 300$ or $300 < x$. When $50 \le x \le 300$, the charge is $.06x$ dollars. When $300 < x$, the charge is $.02x + 12$. The domain consists of the values of x in one of the two intervals $[50, 300]$ and $(300, \infty)$. In each of these intervals, the function is defined by a separate formula:

$$f(x) = \begin{cases} .06x & \text{for } 50 \le x \le 300 \\ .02x + 12 & \text{for } 300 < x. \end{cases}$$

Note that an alternative description of the domain is the interval $[50, \infty)$. That is, the value of x may be any real number greater than or equal to 50.

(b) Since $x = 100$ satisfies $50 \le x \le 300$, we use the first formula for $f(x)$: $f(100) = .06(100) = 6$. Since $x = 500$ satisfies $300 < x$, we use the second formula for $f(x)$: $f(500) = .02(500) + 12 = 22$.

Now Try Exercise 57

In calculus, it is often necessary to substitute an algebraic expression for x and simplify the result, as illustrated in the following example.

EXAMPLE 6 **Evaluating a Function** If $f(x) = (4 - x)/(x^2 + 3)$, what is $f(a)$? $f(a + 1)$?

SOLUTION Here, a represents some number. To find $f(a)$, we substitute a for x wherever x appears in the formula defining $f(x)$:

$$f(a) = \frac{4 - a}{a^2 + 3}.$$

To evaluate $f(a+1)$, substitute $a+1$ for each occurrence of x in the formula for $f(x)$:

$$f(a + 1) = \frac{4 - (a + 1)}{(a + 1)^2 + 3}.$$

We can simplify the expression for $f(a+1)$ using the fact that $(a+1)^2 = (a+1)(a+1) = a^2 + 2a + 1$:

$$f(a + 1) = \frac{4 - (a + 1)}{(a + 1)^2 + 3} = \frac{4 - a - 1}{a^2 + 2a + 1 + 3} = \frac{3 - a}{a^2 + 2a + 4}.$$

More about the Domain of a Function When defining a function, it is necessary to specify the domain of the function, which is the set of acceptable values of the variable. In the preceding examples, we explicitly specified the domains of the functions considered. However, throughout the remainder of the text, we will usually mention functions without specifying domains. In such circumstances, we will understand the intended domain to consist of all numbers for which the defining formula(s) makes sense. For example, consider the function

$$f(x) = x^2 - x + 1.$$

The expression on the right may be evaluated for any value of x. So, in the absence of any explicit restrictions on x, the domain is understood to consist of all numbers. As a second example, consider the function

$$f(x) = \frac{1}{x}.$$

Here x may be any number except zero. (Division by zero is not permissible.) So the domain intended is the set of nonzero numbers. Similarly, when we write

$$f(x) = \sqrt{x},$$

we understand the domain of $f(x)$ to be the set of all nonnegative numbers, since the square root of a number x is defined if and only if $x \geq 0$.

EXAMPLE 7 **Domains of Functions** Find the domains of the following functions:

(a) $f(x) = \sqrt{4+x}$ **(b)** $g(x) = \dfrac{1}{\sqrt{1+2x}}$ **(c)** $h(x) = \sqrt{1+x} - \sqrt{1-x}$

SOLUTION

(a) Since we cannot take the square root of a negative number, we must have $4 + x \geq 0$, or equivalently, $x \geq -4$. So the domain of f is $[-4, \infty)$.

(b) Here, the domain consists of all x for which

$$\begin{aligned} 1 + 2x &> 0 \\ 2x &> -1 && \text{Simplifying} \\ x &> -\frac{1}{2} && \text{Dividing both sides by 2.} \end{aligned}$$

The domain is the open interval $(-\frac{1}{2}, \infty)$.

(c) In order to be able to evaluate both square roots that appear in the expression of $h(x)$, we must have

$$1 + x \geq 0 \quad \text{and} \quad 1 - x \geq 0.$$

The first inequality is equivalent to $x \geq -1$, and the second one to $x \leq 1$. Since x must satisfy both inequalities, it follows that the domain of h consists of all x for which $-1 \leq x \leq 1$.

▸ ***Now Try Exercise 25***

Graphs of Functions

Often it is helpful to describe a function f geometrically, using a rectangular xy-coordinate system. Given any x in the domain of f, we can plot the point $(x, f(x))$. This is the point in the xy-plane whose y-coordinate is the value of the function at x. The set of *all* such points $(x, f(x))$ usually forms a curve in the xy-plane and is called the *graph of the function* $f(x)$.

It is possible to approximate the graph of $f(x)$ by plotting the points $(x, f(x))$ for a representative set of values of x and joining them by a smooth curve. (See Fig. 3.) The more closely spaced the values of x, the closer the approximation.

EXAMPLE 8 **Sketching a Graph by Plotting Points** Sketch the graph of the function $f(x) = x^3$.

SOLUTION The domain consists of all numbers x. We choose some representative values of x and tabulate the corresponding values of $f(x)$. We then plot the points $(x, f(x))$ and draw a smooth curve through the points. (See Fig. 4.)

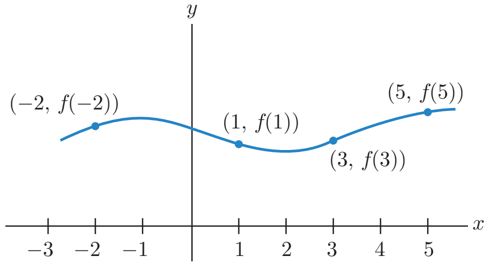


Figure 3

x	$f(x) = x^3$
0	0
1	1
2	8
3	27
−1	−1
−2	−8
−3	−27

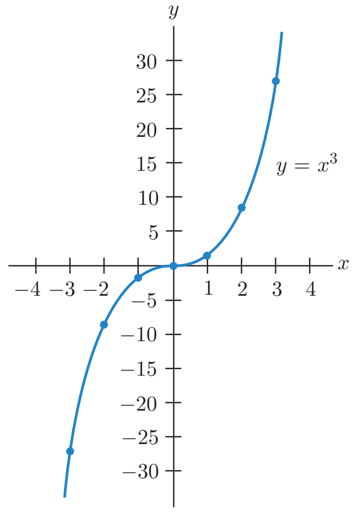


Figure 4 Graph of $f(x) = x^3$.

EXAMPLE 9 **A Graph with a Restricted Domain** Sketch the graph of the function $f(x) = 1/x$.

SOLUTION The domain of the function consists of all numbers except zero. The table in Fig. 5 lists some representative values of x and the corresponding values of $f(x)$. A function often has interesting behavior for x near a number not in the domain. So, when we chose representative values of x from the domain, we included some values close to zero. The points $(x, f(x))$ are plotted and the graph sketched in Fig. 5.

x	$f(x) = \frac{1}{x}$
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$

x	$f(x) = \frac{1}{x}$
$-\frac{1}{4}$	−4
$-\frac{1}{2}$	−2
−1	−1
−2	$-\frac{1}{2}$
−3	$-\frac{1}{3}$
−4	$-\frac{1}{4}$

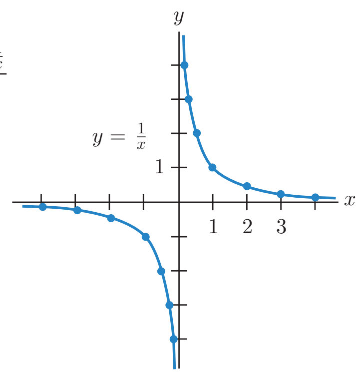


Figure 5 Graph of $f(x) = \frac{1}{x}$.

Now that graphing calculators and computer graphing programs are widely available, we seldom need to sketch graphs by hand-plotting large numbers of points on graph paper. However, to use such a calculator or program effectively, we must know in advance which part of a curve to display. Critical features of a graph may be missed or misinterpreted if, for instance, the scale on the x- or y-axis is inappropriate.

An important use of calculus is to identify key features of a function that should appear in its graph. In many cases, only a few points need be plotted, and the general shape of the graph is easy to sketch by hand. For more complicated functions, a graphing program is helpful. Even then, calculus provides a way of checking that the graph on the computer screen has the correct shape. Algebraic calculations are usually part of the analysis. The appropriate algebraic skills are reviewed in this chapter.

Note also that analytic solutions to problems typically provide more precise information than graphing calculators and can provide insight into the behavior of the functions involved in the solution.

The connection between a function and its graph is explored in this section and in Section 0.6.

EXAMPLE 10

Reading a Graph Suppose that f is the function whose graph is given in Fig. 6. Notice that the point $(x, y) = (3, 2)$ is on the graph of f.

(a) What is the value of the function when $x = 3$?

(b) Find $f(-2)$.

(c) What is the domain of f? What is its range?

SOLUTION

(a) Since $(3, 2)$ is on the graph of f, the y-coordinate 2 must be the value of f at the x-coordinate 3. That is, $f(3) = 2$.

(b) To find $f(-2)$, we look at the y-coordinate of the point on the graph where $x = -2$. From Fig. 6, we see that $(-2, 1)$ is on the graph of f. Thus, $f(-2) = 1$.

(c) The points on the graph of $f(x)$ all have x-coordinates between -3 and 5 inclusive; and for each value of x between -3 and 5, there is a point $(x, f(x))$ on the graph. So the domain consists of those x for which $-3 \le x \le 5$. From Fig. 6, the function assumes all values between .2 and 2.5. Thus, the range of f is [.2, 2.5].

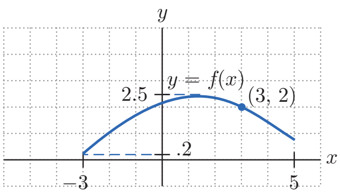


Figure 6

▶ *Now Try Exercise 37*

As we saw in Example 10, the graph of f can be used to picture the domain of f on the x-axis and its range on the y-axis. The general situation is illustrated in Fig. 7.

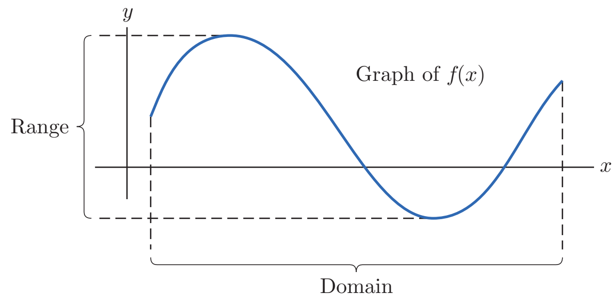


Figure 7 Domain and range.

To every x in the domain, a function assigns one and only one value of y, that is, the function value $f(x)$. This implies, among other things, that not every curve is the graph of a function. To see this, refer first to the curve in Fig. 6, which *is* the graph of a function. It has the following important property: For each x between -3 and 5 inclusive, there is a *unique* y such that (x, y) is on the curve. The variable y is called the *dependent variable*, since its value depends on the value of the independent variable x. Refer to the curve in Fig. 8. It cannot be the graph of a function because a function f must assign to each x in its domain a *unique* value $f(x)$. However, for the curve of Fig. 8, there corresponds to $x = 3$ (for example) more than one y-value: $y = 1$ and $y = 4$.

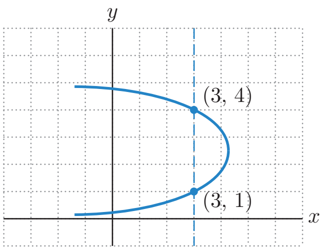


Figure 8 A curve that is *not* the graph of a function.

The essential difference between the curves in Figs. 6 and 8 leads us to the following test.

The Vertical Line Test A curve in the xy-plane is the graph of a function if and only if each vertical line cuts or touches the curve at no more than one point.